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The $\hat{G}^4\lambda^{16}$ term in IIB supergravity

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Abstract: The supersymmetry constraints on the $\hat{G}^4\lambda^{16}$ term in the effective action of type IIB superstring theory are studied in order to determine the dependence of its coefficient on the complex scalar field, τ. The resulting expression is consistent with the $\mathrm{SL}(2,\mathbb{Z})$ invariant conjectures in the literature.

Keywords: Superstrings and Heterotic Strings, Supersymmetric Effective Theories.

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1. Introduction

Chiral $N = 2$, $D = 10$ supergravity [1, 2] is the low energy limit of type IIB string theory. Higher derivative terms in the low energy limit can be generated by considering scattering amplitudes in string perturbation theory. This determines the terms proportional to $e^{-2\phi}$, where ϕ is the dilaton. However, determining the exact dependence of these terms on the scalar fields is more challenging. In principle, perturbative contributions can be determined for higher genus string loop calculations, but there is no direct way of determining the non-perturbative contributions. The exact action must be invariant under $\mathrm{SL}(2,\mathbb{Z})$ which means that the scalar field dependence is encoded in modular forms, depending on $\tau = C^0 + ie^{-\phi}$, where C^0 is the Ramond-Ramond scalar.

The constraints imposed by supersymmetry are very powerful. In [3, 4], it was shown how to use supersymmetry to compute the coefficient for the 16-dilatino term which appears at eight-derivative order, i.e., at α'^3 relative to the tree-level. This term has been analyzed in [5]. The coefficient turns out to satisfy an eigen-value equation for the laplacian on the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. The solution for such an equation is a generalized Eisenstein series [6, 4, 7]. This series has an expansion which encodes tree-level and higher genus information along with an infinite series of D-instanton contributions. The 16-dilatino term is related by linearized supersymmetry to the C^4 term, where C denotes the Weyl tensor and C^4 symbolizes the contraction of four of these. The coefficient of this term is a function of $\tau, \bar{\tau}$ which has been studied through different consistent arguments [8]–[12], which serve as a powerful countercheck for the validity of the calculation. The coefficient implies that the C^4 term gets only tree-level, one-loop and a series of D-instanton contributions. The validity of this powerful prediction has been checked by explicit two-loop calculation for the four-graviton in [13, 14]. It was shown that there is no genus-2 contribution. Furthermore, a non-renormalization theorem was proved in [15] which showed C^4 cannot receive perturbative contributions beyond tree-level and one-loop.

There have been other ways to infer the coefficients of higher-derivative terms. In [17], an all genus conjecture for terms like $C^4\hat{G}^{4g-4}$ in type IIB in ten dimensions was made, where $\hat{G}$ stands for the supercovariant antisymmetric three-form field strength and g is the genus. The argument was motivated by $N = 4$ topological string theory [16]. Strong evidence was presented that the coefficient of these terms are higher order Eisenstein series. All these Eisenstein series have the generic feature of representing tree-level and genus-g contributions as well as a series of D-instanton contributions. It should be possible to prove these conjectures by using the supersymmetric methods of [3].

Since the C^4 interaction is related by superspace arguments to the 16-dilatino interaction, it is expected that there will be $\hat{G}^4\lambda^{16}$ term in the action, λ being the dilatino, at the twelve-derivative or order α'^5 relative to the tree-level terms.

In [3], motivated by [17, 18], a conjecture was made for higher derivative extension of the IIB effective action. It reads

$$(\alpha')^4 \sum_{g,\hat{g}=1}^{\infty} \sum_{p=2-2g}^{2g-2} (\alpha')^{2g+2\hat{g}-1} \int d^{10}x \, \det e F_5^{4\hat{g}-4} \hat{G}^{2g-2+p} \hat{G}^{*2g-2-p} \times \\ \times \left(f_{g+\hat{g}-1}^{(p,-p)}(\tau,\bar{\tau}) C^4 + \cdots + f_{g+\hat{g}-1}^{(12+p,-12-p)}(\tau,\bar{\tau}) \lambda^{16} \right), \tag{1.1}$$

where F_5 is the self-dual 5-form field strength and $\hat{G}$, the supercovariant version of the field strength G is

$$\hat{G}_{\mu\nu\rho} = G_{\mu\nu\rho} - 3\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\lambda - 6i\bar{\psi}^*{}_{[\mu}\gamma_\nu\psi_{\rho]}\,, \tag{1.2}$$

where ψ_μ is the gravitino. The modular forms $f_g^{(q,-q)}$ are expected to be given by the generalized Eisenstein series

$$f_g^{(q,-q)} = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^{g+\frac{1}{2}}}{(m+n\tau)^{g+\frac{1}{2}+q}(m+n\bar{\tau})^{g+\frac{1}{2}-q}}\,. \tag{1.3}$$

For $q = 0$ these functions are proportional to $E_{g+\frac{1}{2}}$ where E_s is defined in equation (B.12). For $g = 2$, $\hat{g} = 1$, $p = 2$ there is evidently a $\det e\hat{G}^4\lambda^{16}$ term in the integrand. The coefficient of this term is conjectured to be $f_2^{(14,-14)}$. In [3], a schematic method of obtaining the coefficient of this term was presented using supersymmetry arguments. However, the calculation was not completed and though it seemed plausible, the fact that the coefficient of such a term is a generalized Eisenstein series, was not proved. Such a proof will give further evidence for the conjectures in the literature for the ten-dimensional effective action. Implications using the AdS/CFT correspondence for this term are also currently being investigated [19].

In this paper, we construct a proof using supersymmetry that the coefficient of the $G^4\lambda^{16}$ term is the expected modular form derived from the Eisenstein series, $E_{5/2}$.

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The paper is organized as follows. In section 2, we outline the method used in [3] to obtain the coefficient of the 16-dilatino term. In section 3, we determine the coefficient for $\hat{G}^4\lambda^{16}$.In section 4, we discuss the tensor structure for $\hat{G}^4\lambda^{16}$. Two appendices have been included which summarize various identities and supersymmetry transformations required in the paper.

2. Review of λ^{16} term at order α'^3

In this section we briefly summarize the supersymmetry calculation at order α'^3 as done in [3]. The notation is made clear in appendix A. There is no off-shell superspace formulation for the theory as a result of which an action with manifest supersymmetry cannot be written down. It is possible to write on-shell superfields [20] and use them to write manifestly supersymmetric equations of motion. In what follows the lagrangian will just be a shorthand for the equations of motion. The low-energy effective action can be written as

$$S = \int d^{10}x\sqrt{g}L \sim \frac{1}{\alpha'^4}\left[S^{(0)} + \alpha'^3 S^{(3)} + \alpha'^4 S^{(4)} + \alpha'^5 S^{(5)} + \ldots\right], \tag{2.1}$$

where L is the lagrangian and there are no contributions at α' and α'^2 order. The supersymmetry transformation δ can also be expanded in powers of α'.

$$\delta \sim \delta^{(0)} + \alpha'^3\delta^{(3)} + \alpha'^4\delta^{(4)} + \alpha'^5\delta^{(5)} + \ldots. \tag{2.2}$$

The Noether method of constructing supersymmetric actions demands that the supersymmetry transformations close on using the equations of motion. This will yield supersymmetry constraints of the form

$$\delta^{(0)}S^{(0)} = \delta^{(0)}S^{(3)} + \delta^{(3)}S^{(0)} = \delta^{(0)}S^5 + \delta^{(5)}S^{(0)} = \cdots = 0\,. \tag{2.3}$$

The calculation proceeds as follows. Two specific terms in the effective lagrangian are selected which do not mix under supersymmetry with any other terms at this order. These are:

$$L_1^{(3)} = \det e\left(f^{(12,-12)}(\tau,\bar\tau)\lambda^{16} + f^{(11,-11)}(\tau,\bar\tau)(\lambda^{15}\gamma^\mu\psi_\mu^*)\right). \tag{2.4}$$

The normalization of the terms have been changed slightly from that in [3] for our convenience. In type IIB supergravity, we define two supersymmetry parameters ϵ and ϵ^*. The lowest order ϵ supersymmetry transformations of $L_1^{(3)}$ contain a term proportional to $\det e\lambda^{16}\psi_\mu^*\epsilon$ with a coefficient that has to vanish for the action to be supersymmetric, leading to the condition

$$\mathrm{D}_{11}f^{(11,-11)}(\tau,\bar\tau) = 4f^{(12,-12)}(\tau,\bar\tau)\,, \tag{2.5}$$

where the notation D_{11} is explained in appendix A. The ϵ^* variation of (2.4) gives a term of the form $\det e\lambda^{16}\lambda^*\epsilon^*$. In this case, equation (2.3) can only be satisfied if account is taken of a term from the lowest order IIB lagrangian,

$$L_1^{(0)} = \frac{1}{256}\det e\bar\lambda\gamma^{\rho_1\rho_2\rho_3}\lambda^*\bar\lambda^*\gamma_{\rho_1\rho_2\rho_3}\lambda\,. \tag{2.6}$$

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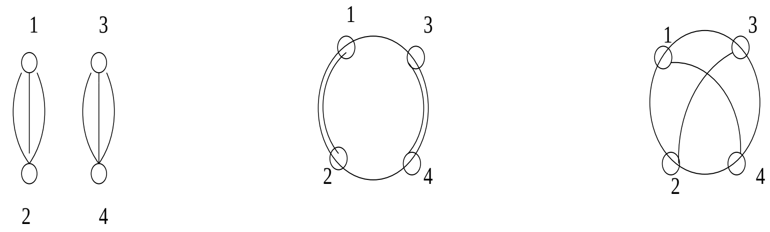

(a) (b) (c)

Figure 1: Diagrammatic representation of the various contractions. (a) represents structure T_3 in the paper, (b) represents T_1 and (c) represents T_2. The lines indicate contractions and the small circles represent $\hat{G}$.

A modification to the ϵ^* supersymmetry transformation of λ^*, of the form

$$\delta_1^{(3)}\lambda_a^* = g(\tau,\bar{\tau})\hat{G}^4(\lambda^{14})_{cd}(\gamma^{\mu\nu\rho}\gamma^0)_{dc}(\gamma_{\mu\nu\rho}\epsilon^*)_a\,, \tag{2.7}$$

where $g(\tau,\bar{\tau})$ is an unknown function, acting on $L_1^{(0)}$ leads to

$$2\bar{\mathrm{D}}_{-12}f^{(12,-12)}(\tau,\bar{\tau}) + 15f^{(11,-11)}(\tau,\bar{\tau}) - 3\cdot 360ig = 0\,. \tag{2.8}$$

Finally, we obtain a constraint by demanding the closure of the supersymmetry algebra on λ^*. This gives rise to the relation:

$$-192i\mathrm{D}_{11}g = f^{(12,-12)}(\tau,\bar{\tau}) \tag{2.9}$$

Combining (2.5), (2.8), (2.9) gives

$$\nabla^2_{(-)12}f^{(12,-12)}(\tau,\bar{\tau}) = \left(-132 + \frac{3}{4}\right)f^{(12,-12)}(\tau,\bar{\tau})\,. \tag{2.10}$$

This is exactly what we expect as shown in equation (B.10) in the appendix.

3. $\hat{G}^4\lambda^{16}$ term at order α'^5

By considering the generalization of the preceding argument to the α'^5 term $\hat{G}^4\lambda^{16}$, one can show that its coefficient is a modular form. The term involving gravitinos in $\hat{G}^4\lambda^{16}$ is schematically $(\psi\psi)^4\lambda^{16}$. Since λ^{16} forms a Lorentz singlet, $\hat{G}^4$ will also be a Lorentz singlet. In principle, there are three independent ways to form a singlet using four $\hat{G}$s, which are diagrammatically represented in figure 1.

In terms of space-time indices, the contractions are given by

$$\begin{aligned}\hat{G}^4 = {} & a\,\hat{G}_{\mu_1\nu_1\rho_1}\hat{G}^{\mu_1\mu_2\rho_1}\hat{G}_{\mu_3}{}^{\nu_1\nu_2}\hat{G}_{\mu_2\nu_2}{}^{\mu_3} + b\,\hat{G}_{\mu_1\nu_1\rho_1}\hat{G}^{\mu_1\mu_3\nu_2}\hat{G}_{\mu_3\mu_4}{}^{\rho_1}\hat{G}^{\mu_4\nu_1}{}_{\nu_2} + \\ & + c\,\hat{G}_{\mu_1\nu_1\rho_1}\hat{G}^{\mu_1\nu_1\rho_1}\hat{G}_{\mu_2\nu_2\rho_2}\hat{G}^{\mu_2\nu_2\rho_2}\,,\end{aligned} \tag{3.1}$$

where a, b, c are undetermined coefficients which are assumed to be non-zero. The following argument will not yield the values of a, b, c. To be very specific, the term proportional to

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$\hat{G}_{\mu_1\nu_1\rho_1}\hat{G}^{\mu_1\nu_1\rho_1}\hat{G}_{\mu_2\nu_2\rho_2}\hat{G}^{\mu_2\nu_2\rho_2} \sim \hat{G}^2\hat{G}^2$ will be considered, though it is easy to generalize the argument to the other two cases. The piece of $\hat{G}_{\mu\nu\rho}$ involving gravitino bilinears will be written using the shorthand notation $(-6i\psi\psi)^4$.

Following [3] we will now select three terms with the appropriate dimensions contributing to $S^{(5)}$ that will mix with each other, but with no other terms, under supersymmetry. These are

$$L_1^{(5)} = \det e\lambda^{16}\hat{G}^4 f_2^{(14,-14)} \tag{3.2}$$
$$L_2^{(5)} = \det e\lambda^{15}\gamma^\mu\psi^*_\mu\hat{G}^4 f_2^{(13,-13)} \tag{3.3}$$
$$L_3^{(5)} = \det e\lambda^{16}\hat{G}^2\hat{G}_{\rho_1\rho_2\rho_3}\hat{G}^{*\rho_1\rho_2\rho_3}\tilde{f}_2^{(13,-13)}\,. \tag{3.4}$$

The ϵ^* supersymmetry variation proportional to $\det e\lambda^{16}\bar{\epsilon}\lambda^*(-6i\psi\psi)^4$ gives

$$\begin{aligned}\delta_1^{(0)}L_1^{(5)} &= -2\det e\lambda^{16}(\bar{\epsilon}\lambda^*)(-6i\psi\psi)^4\left(\tau_2\frac{\partial}{\partial\tau} - 7i\right)f_2^{(14,-14)} \\ &= -2i\det e\lambda^{16}\bar{\epsilon}\lambda^*(-6i\psi\psi)^4\bar{\mathrm{D}}_{-14}f_2^{(14,-14)}\,,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\delta_1^{(0)}L_2^{(5)} &= \det e\lambda^{15}\delta(\gamma^\mu\psi^*_\mu)(-6i\psi\psi)^4 f_2^{(13,-13)} \\ &= -15i\det e\lambda^{16}\bar{\epsilon}\lambda^*(-6i\psi\psi)^4\tilde{f}_2^{(13,-13)}\,.\end{aligned} \tag{3.6}$$

$\hat{G}^*$ in $L_3^{(5)}$ has a term of the form $\psi_\mu\lambda^*$. Taking into account the fact that the ϵ^* supersymmetry variation of ψ_μ, as given in appendix A, has a $\hat{G}$ piece and the ϵ^* variation of λ^* has a $\psi_\mu\lambda^*$ piece, the following equation is obtained

$$\delta_1^{(0)}L_3^{(5)} = -\frac{9}{16}\tilde{f}_2^{(13,-13)}\det e\lambda^{16}(-6i\psi\psi)^4\bar{\epsilon}\lambda^*\,. \tag{3.7}$$

In addition we now consider the $O(\alpha'^5)$ supersymmetry transformations acting on the following two terms from the classical action,

$$L_1^{(0)} = \frac{1}{256}\det e\bar{\lambda}\gamma^{\rho_1\rho_2\rho_3}\lambda^*\bar{\lambda^*}\gamma_{\rho_1\rho_2\rho_3}\lambda \tag{3.8}$$
$$L_2^{(0)} = -\frac{1}{8}\bar{\psi}^*_\mu\gamma_{\nu\rho}\lambda^*\hat{G}^{\mu\nu\rho}\,, \tag{3.9}$$

where $L_1^{(0)}$ is the same interaction considered in the previous section and $L_2^{(0)}$ can be read off from [1, equation (4.12)]. The modified ϵ^* supersymmetry transformations at order α'^5 are

$$\delta_1^{(5)}\lambda^*_a = g_1(\tau,\bar{\tau})\hat{G}^4(\lambda^{14})_{cd}(\gamma^{\mu\nu\rho}\gamma^0)_{dc}(\gamma_{\mu\nu\rho}\epsilon^*)_a \tag{3.10}$$
$$\delta_1^{(5)}\psi_{\mu_a} = g_2(\tau,\bar{\tau})\lambda^{16}\hat{G}^2\hat{G}_{\rho_1\rho_2\rho_3}(\gamma^{\rho_1\rho_2\rho_3}\gamma_\mu)_{ab}\epsilon^*_b\,, \tag{3.11}$$

where g_1 and g_2 are unknown functions of τ and $\bar{\tau}$. These transformations acting on $L_1^{(0)}$ and $L_2^{(0)}$ give

$$\delta_1^{(5)}L_1^{(0)} = -3.360\det e\, g_1\hat{G}^4\lambda^{16}\bar{\epsilon}\lambda^* \tag{3.12}$$
$$\delta_1^{(5)}L_2^{(0)} = -\frac{3}{4}\det e\, g_2\hat{G}^4\lambda^{16}\bar{\epsilon}\lambda^*\,. \tag{3.13}$$

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In order to satisfy the constraint (2.3), the following equation is obtained

$$2\bar{\mathrm{D}}_{-14} f_2^{(14,-14)} + 15 f_2^{(13,-13)} - \frac{9i}{16}\tilde{f}_2^{(13,-13)} - 3\cdot 360 i\, g_1 - \frac{3}{4} i\, g_2 = 0\,. \tag{3.14}$$

Now we consider supersymmetry variations of the form $(\det e\lambda^{16}(-6i\psi\psi)^4)\bar{\epsilon^*}\gamma^\mu\psi^*_\mu$. The term $L_3^{(5)}$ doesn't mix since $\hat{G}^*$ has either a $\psi^*_\mu\psi^*_\nu$ piece or a $\psi_\mu\lambda^*$ piece, neither of which yield $\psi^*\psi\psi$ under supersymmetry variation. Variation of $L_1^{(5)}$ yields,

$$\begin{aligned}\delta_2^{(0)} L_1^{(5)} &= (\delta_2^{(0)} \det e)\lambda^{16}(-6i\psi\psi)^4 f_2^{(14,-14)} + \det e\,(\delta_2^{(0)}\lambda^{16})(-6i\psi\psi)^4 f_2^{(14,-14)} + \\ &\quad + \det e\lambda^{16}\left(\delta_2^{(0)}(-6i\psi\psi)^4\right) f_2^{(14,-14)}\,.\end{aligned} \tag{3.15}$$

Here the first two terms are given in [3, equation (3.4)]. Together they give $(2i\mathrm{D}_{13} f_2^{(13,-13)} - 8i f_2^{(14,-14)}) \det e\, \bar{\epsilon^*}\gamma^\mu\psi^*_\mu\lambda^{16}$. The last term can be written as:

$$\det e\lambda^{16}(\delta_2^{(0)}(-6i\psi\psi)^4) f_2^{(14,-14)} = \left(-\frac{7}{4} - \frac{5}{4}\right) i \det e\lambda^{16}(-6i\psi\psi)^4\bar{\epsilon^*}\gamma^\mu\psi^*_\mu\,, \tag{3.16}$$

where the first term in the bracket on the right comes from the $\hat{F}_5$ in the variation of ψ_μ and the second comes from the supercovariant derivative acting on ϵ. The supercovariant derivative D_μ has a piece that depends on the gravitino bilinear [1] which has been taken into account. Thus,

$$\delta_2^{(0)} L_1^{(5)} = \left(2\mathrm{D}_{13} f_2^{(13,-13)} - \left(8 + \frac{7}{4} + \frac{5}{4}\right) f_2^{(14,-14)}\right) i \det e\lambda^{16}(-6i\psi\psi)^4\bar{\epsilon^*}\gamma^\mu\psi^*_\mu\,, \tag{3.17}$$

and the action is supersymmetric if

$$2\mathrm{D}_{13} f_2^{(13,-13)} - \left(8 + \frac{7}{4} + \frac{5}{4}\right) f_2^{(14,-14)} = 0\,. \tag{3.18}$$

Further constraints are imposed by demanding the closure of the supersymmetry algebra. In particular $[\delta_{\epsilon_1}, \delta_{\epsilon_2^*}]\lambda^*$ and $[\delta_{\epsilon_1}, \delta_{\epsilon_2^*}]\psi_\mu$ are considered. In a manner similar to deriving (3.17) of [3] we get

$$\begin{aligned}[\delta^0_{\epsilon_1}, \delta^5_{\epsilon_2^*}]\lambda^*_a &= -192 i\mathrm{D}_{13} g_1 \lambda^{15}_b \left[\frac{3}{8}\bar{\epsilon}_2\gamma^\sigma\epsilon_1(\gamma_\sigma)_{ba}\right] \\ &\quad + g_1(\delta^0_{\epsilon_1}\hat{G}^4)(\lambda^{14})_{cd}(\gamma^{\mu\nu\rho}\gamma^0)_{dc}(\gamma_{\mu\nu\rho}\epsilon_2^*)_a + \cdots\,,\end{aligned} \tag{3.19}$$

where the ellipsis indicate terms that are not needed for the analysis. To evaluate the last term note that $\delta\hat{G} \sim \delta(\psi\psi) + \delta(\psi^*\lambda)$. $\delta\psi^*$ will have a $\hat{G}^*$ piece and this will relate g_1 to $\tilde{f}_2^{(13,-13)}$. The result is

$$g_1(\delta^0_{\epsilon_1}\hat{G}^4)(\lambda^{14})_{cd}(\gamma^{\mu\nu\rho}\gamma^0)_{dc}(\gamma_{\mu\nu\rho}\epsilon_2^*)_a = -108 g_1\hat{G}^2(\hat{G}^{\mu\nu\lambda}\hat{G}^*_{\mu\nu\lambda})\lambda^{15}_b\left(\frac{3}{8}\bar{\epsilon}_2\gamma^\sigma\epsilon_1(\gamma_\sigma)_{ba}\right) + \cdots \tag{3.20}$$

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Thus the commutator of δ_{ϵ_1} and $\delta_{\epsilon_2^*}$ acting on λ^* is

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2^*}]\lambda_a^* = \bar{\epsilon_2}\gamma^\mu\epsilon_1 \mathrm{D}_\mu \lambda_a^* + +\frac{3}{8}\bar{\epsilon_2}\gamma^\sigma\epsilon_1(\gamma_\sigma)_{ba}\times \tag{3.21}$$
$$\times \left\{-i(\gamma^\mu \mathrm{D}_\mu\lambda^*)_b + \alpha'^5(-192i\mathrm{D}_{13}g_1\lambda_b^{15} - 108 g_1 \hat{G}^2(\hat{G}^{\mu\nu\lambda}\hat{G}^*_{\mu\nu\lambda})\lambda_b^{15})\right\} + \cdots ,$$

where the first two terms on the right-hand-side come from the commutator of the lowest order supersymmetry transformations [1, 3]. In order to close the algebra, the equations of motion have to be used. These give

$$-192i\mathrm{D}_{13}g_1 = f_2^{(14,-14)} \tag{3.22}$$
$$-108 g_1 = \tilde{f}_2^{(13,-13)} . \tag{3.23}$$

Terms proportional to $(\epsilon_1\lambda^{15}\hat{G}^4\epsilon_2^*)_\mu$ in $[\delta_{\epsilon_1}, \delta_{\epsilon_2^*}]\psi_\mu$ are now considered. Using various Fierz transformations and gamma product expansions we find

$$\delta^0_{\epsilon_1}\delta^5_{\epsilon_2^*}(\psi_\mu)_c = \frac{i}{64}g_2\lambda_a^{15}\hat{G}^4(\gamma_\mu\gamma^\sigma)_{ac}(\bar{\epsilon_2}\gamma_\sigma\epsilon_1) + \cdots . \tag{3.24}$$

In addition, $\delta^5_{\epsilon_2^*}\delta^0_{\epsilon_1}\psi_\mu$ contributions proportional to $\lambda_a^{15}\hat{G}^4(\gamma_\mu\gamma^\sigma)_{ac}(\bar{\epsilon_2}\gamma_\sigma\epsilon_1)$ also need to be taken into account. $\delta^{(0)}\psi_\mu$ has terms proportional to $\lambda\lambda^*$ (See equation (A.6) in the appendix). These come from the $\hat{F}_5$ piece as well as the other $\lambda\lambda^*$ terms in $\delta\psi_\mu$. The relevant terms in the supersymmetry transformation for ψ_μ proportional to ϵ are

$$\begin{aligned}\delta^0_{\epsilon_1}\psi_\mu = & -\frac{i}{480.16}\gamma^{\rho_1\ldots\rho_5}\gamma_\mu\epsilon_1\bar{\lambda}\gamma_{\rho_1\ldots\rho_5}\lambda + \\ & +\frac{i}{32}\left[\left(\frac{9}{4}\gamma_\mu\gamma^\rho + 3\gamma^\rho\gamma_\mu\right)\epsilon_1\bar{\lambda}\gamma_\rho\lambda - \left(\frac{1}{24}\gamma_\mu\gamma^{\rho_1\rho_2\rho_3} + \frac{1}{6}\gamma^{\rho_1\rho_2\rho_3}\gamma_\mu\right)\epsilon_1\bar{\lambda}\gamma_{\rho_1\rho_2\rho_3}\lambda + \right.\\ & \left. + \frac{1}{960}\gamma_\mu\gamma^{\rho_1\ldots\rho_5}\epsilon_1\bar{\lambda}\gamma_{\rho_1\ldots\rho_5}\lambda\right] + \cdots ,\end{aligned} \tag{3.25}$$

where the first term comes from $\hat{F}_5$. The contribution of the first term to $\delta^5_{\epsilon_2^*}\delta^0_{\epsilon_1}\psi_\mu$ is

$$\frac{49}{64}ig_1\hat{G}^4\lambda_d^{15}(\gamma_\mu\gamma^\sigma)_{ad}\bar{\epsilon_2}\gamma_\sigma\epsilon_1 . \tag{3.26}$$

The contribution to $\delta^5_{\epsilon_2^*}\delta^0_{\epsilon_1}\psi_\mu$ from the term in equation (3.25) proportional to $\bar{\lambda}\gamma_\rho\lambda$ can be shown to vanish. The term proportional to $\bar{\lambda}\gamma^{\rho_1\rho_2\rho_3}\lambda$ in equation (3.25) yields the following contribution to $\delta^5_{\epsilon_2^*}\delta^0_{\epsilon_1}\psi_\mu$

$$-\left(\frac{9}{4} + \frac{3}{2}\right)ig_1\hat{G}^4\lambda_d^{15}(\gamma_\mu\gamma^\sigma)_{ad}\bar{\epsilon_2}\gamma_\sigma\epsilon_1 . \tag{3.27}$$

Using $\gamma^{\rho_1\cdots\rho_5}\gamma^\mu\gamma_{\rho_1\ldots\rho_5} = 0$, we can show that the term proportional to $\bar{\lambda}\gamma_{\rho_1\ldots\rho_5}\lambda$ does not contribute. Thus we get

$$\delta^5_{\epsilon_2^*}\delta^0_{\epsilon_1}\psi_{\mu a} = -\frac{191}{64}ig_1\hat{G}^4\lambda_d^{15}(\gamma_\mu\gamma^\sigma)_{ad}\bar{\epsilon_2}\gamma_\sigma\epsilon_1 . \tag{3.28}$$

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Demanding the closure of the supersymmetry algebra on using the equations of motion gives

$$ig_2 + 191ig_1 = \frac{1}{2}f_2^{(13,-13)} . \tag{3.29}$$

In order to derive the above result, one needs the following equation which can be obtained by considering the lowest order supersymmetry transformation on the gravitino,

$$\left[\delta^0_{\epsilon_1}, \delta^0_{\epsilon^*_2}\right]\psi_\mu = -\frac{i}{64}(\bar{\epsilon_2}\gamma_\sigma\epsilon_1)\gamma^\sigma\gamma_\mu{}^{\rho\lambda}\mathrm{D}_\rho\psi_\lambda + \cdots , \tag{3.30}$$

where the ellipsis indicate terms that are not needed in the calculation. Using equations (3.14), (3.18), (3.23) and (3.29) we get:

$$\nabla^2_{(-)14}f_2^{(14,-14)} = \left(-182 + \frac{15}{4}\right)f_2^{(14,-14)} , \tag{3.31}$$

which is what is expected from equation (B.16).

The solution for the equation above is given by $\mathrm{D}^{14}E_{5/2}$ where $E_{5/2}$ is the Eisenstein series of order 5/2. The expansion of the Eisenstein series as given in equation (B.13) suggests that the $\hat{G}^4\lambda^{16}$ term we have been considering receives correction from tree-level and two-loops and a series of non-perturbative D-instanton contributions. This is consistent with the generalized higher-derivative conjectures in the literature [3, 17].

4. Discussion

In this paper we have concluded the supersymmetry argument initiated in [3] for the $\hat{G}^4\lambda^{16}$ term in the type IIB effective action. We have shown that the coefficient of this does satisfy the expected eigen-value equation on the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. This gives further evidence for the conjecture given by equation (1.2). Generalizing such arguments based on supersymmetry for higher order terms seems to get far more tedious owing to the mixing at other orders of α'.

The preceding argument was too crude to distinguish between the three different contractions in equation (3.1). In principle, it should be possible to obtain a superspace formulation of these terms which are given by integration over 3/4 of the Grassmann coordinates. However there is no obvious covariant way of doing this. One can motivate the structure of the contractions in $\hat{G}^4\lambda^{16}$ interaction by reference to the analysis of the $\hat{G}^4C^4$ term in [17]. This starts by considering certain six-dimensional superstring scattering amplitudes which can be expressed as topological computations on the hyper-Kahler compactification manifold [16]. The $\hat{G}^4$ factor in $\hat{G}^4\lambda^{16}$ is necessarily a Lorentz singlet which is not the case in the $\hat{G}^4C^4$ term. However, it can be argued that the singlet part of $\hat{G}^4$ contraction in [17] is identical to the $\hat{G}^4$ factor in $\hat{G}^4\lambda^{16}$. As a result it is very suggestive that the $\hat{G}^4$ factor is given by

$$\hat{G}^4 = a\left(T_1 + 15(T_2 - 6T_3)\right) , \tag{4.1}$$

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where

$$T_1 = -G_{\mu_1\nu_1\rho_1}G^{\mu_1\mu_2\rho_1}G_{\mu_3}{}^{\nu_1\nu_2}G_{\mu_2\nu_2}{}^{\mu_3} \tag{4.2}$$

$$T_2 = G_{\mu_1\nu_1\rho_1}G^{\mu_1\mu_3\nu_2}G_{\mu_3\mu_4}{}^{\rho_1}G^{\mu_4\nu_1}{}_{\nu_2} \tag{4.3}$$

$$T_3 = G_{\mu_1\nu_1\rho_1}G^{\mu_1\nu_1\rho_1}G_{\mu_2\nu_2\rho_2}G^{\mu_2\nu_2\rho_2}\,. \tag{4.4}$$

The coefficients in equation (3.1) are thus given by $b = -15a, c = -90a$.

It would be gratifying to have a direct computation in string theory of the D-instanton contributions to the $\hat{G}^4\lambda^{16}$ interaction, but this seems to be problematic at present.

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A. Relevant formulae in IIB supergravity

The bosonic fields of the IIB supergravity comprise of the graviton, the antisymmetric two form with a three-form field strength and the dilaton. The fermionic fields are the gravitino and the dilatino. Spinors in IIB are complex Weyl spinors. The gravitino ψ_μ and the dilatino λ have opposite chiralities, the supersymmetry parameter has the same chirality as the gravitino. The conjugate of any spinor is defined by $\bar{\lambda} = \lambda^*\gamma^0$. The metric is spacelike and the gamma matrices are real. We make extensive use of various identities quoted in [3]. The Fierz identity for ten-dimensional complex Weyl spinors of the same chirality is:

$$\lambda_1^a\bar{\lambda}_2^b = -\frac{1}{16}\bar{\lambda}_2\gamma_\mu\lambda_1\gamma^\mu_{ab} + \frac{1}{96}\bar{\lambda}_2\gamma_{\mu\nu\rho}\lambda_1\gamma^{\mu\nu\rho}_{ab} - \frac{1}{3840}\bar{\lambda}_2\gamma_{\rho_1\ldots\rho_5}\lambda_1\gamma^{\rho_1\ldots\rho_5}_{ab} \tag{A.1}$$

The bosonic fields which appear are supercovariantized in the following way.

$$\begin{aligned}\hat{G}_{\mu\nu\rho} &= G_{\mu\nu\rho} - 3\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\lambda - 6i\bar{\psi}^*{}_{[\mu}\gamma_\nu\psi_{\rho]}\\ \hat{F}_{\rho_1\ldots r5} &= F_{\rho_1\ldots\rho_5} - 5\bar{\psi}_{[\rho_1}\gamma_{\rho_2\rho_3\rho_4}\psi_{\rho_5]} - \frac{1}{16}\bar{\lambda}\gamma_{\rho_1\ldots\rho_5}\lambda\end{aligned} \tag{A.2}$$

The lowest order supersymmetry transformation for the various fields are given below(we retain only the relevant portions, for the complete transformations, see [3, 1]). For τ

$$\delta^{(0)}\tau = 2\tau_2\bar{\epsilon}^*\lambda \tag{A.3}$$

The supersymmetry transformation of the zehnbein is given by:

$$\delta^{(0)}e^m_\mu = i(\bar{\epsilon}\gamma^m\psi_\mu + \text{c.c.}) \tag{A.4}$$

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The transformation for the dilatino in the fixed U(1) gauge is

$$\delta^{(0)}\lambda_a = .. - \frac{i}{24}\gamma^{\mu\nu\rho}\epsilon_a\hat{G}_{\mu\nu\rho} + \frac{3}{4}i\lambda_a(\bar{\epsilon}\lambda^*) - \frac{3}{4}i\lambda_a(\bar{\epsilon}^*\lambda) \tag{A.5}$$

where the last two terms come from the compensating U(1) gauge transformation.

The gravitino transformation is given by

$$\begin{aligned}\delta^{(0)}\psi_\mu = {} & D_\mu\epsilon + \frac{1}{480}i\gamma^{\rho_1\ldots\rho_5}\gamma_\mu\epsilon\hat{F}_{\rho_1\ldots\rho_5} + \frac{1}{96}\left(\gamma_\mu{}^{\nu\rho\lambda}\hat{G}_{\nu\rho\lambda} - 9\gamma^{\rho\lambda}\hat{G}_{\mu\rho\lambda}\right)\epsilon^* - \\ & - \frac{7}{16}\left(\gamma_\rho\lambda\,\bar{\psi}_\mu\gamma^\rho\epsilon^* - \frac{1}{1680}\gamma_{\rho_1\ldots\rho_5}\lambda\,\bar{\psi}_\mu\gamma^{\rho_1\ldots\rho_5}\epsilon^*\right) + \\ & + \frac{1}{32}i\Bigg[\left(\frac{9}{4}\gamma_\mu\gamma^\rho + 3\gamma^\rho\gamma_\mu\right)\epsilon\,\bar{\lambda}\gamma_\rho\lambda - \left(\frac{1}{24}\gamma_\mu\gamma^{\rho_1\rho_2\rho_3} + \frac{1}{6}\gamma^{\rho_1\rho_2\rho_3}\gamma_\mu\right)\epsilon\,\bar{\lambda}\gamma_{\rho_1\rho_2\rho_3}\lambda \\ & + \frac{1}{960}\gamma_\mu\gamma^{\rho_1\ldots\rho_5}\epsilon\,\bar{\lambda}\gamma_{\rho_1\ldots\rho_5}\lambda\Bigg] + \delta^{(0)}_\Sigma(\psi_\mu),\end{aligned} \tag{A.6}$$

where the compensating U(1) transformation is given by

$$\delta^{(0)}_\Sigma\psi_\mu = \frac{1}{2}i\Sigma = \frac{1}{4}i\psi_\mu(\bar{\epsilon}\lambda^*) - \frac{1}{4}i\psi_\mu(\bar{\epsilon}^*\lambda) \tag{A.7}$$

B. Modular covariance and formulae

The various coefficient functions in the effective action are $(w,\hat{w})$ forms, where w refers to the holomorphic modular weight and $\hat{w}$ to the anti-holomorphic modular weight. A nonholomorphic modular form $F^{(w,\hat{w})}$ transforms as,

$$F^{(w,\hat{w})} \to F^{(w,\hat{w})}(c\tau+d)^w(c\bar{\tau}+d)^{\hat{w}} \tag{B.1}$$

under the $SL(2,\mathbb{Z})$ transformation taking,

$$\tau \to \frac{a\tau+b}{c\tau+d} \tag{B.2}$$

where $ad-bc=1$. Equation (B.1) describes a U(1) transformation when $\hat{w}=-w$. We define the modular covariant derivative

$$\mathcal{D}_w = i\left(\frac{\partial}{\partial\tau} - i\frac{w}{2\tau_2}\right) \tag{B.3}$$

This maps $F^{(w,\hat{w})}$ to $F^{(w,\hat{w}+2)}$. We define

$$\mathrm{D}_w \equiv \tau_2\mathcal{D}_w\,, \qquad \bar{\mathrm{D}}_{\hat{w}} \equiv \tau_2\bar{\mathcal{D}}_{\hat{w}} \tag{B.4}$$

This has the effect

$$\mathrm{D}_w F^{(w,\hat{w})} = F^{(w+1,\hat{w}-1)}\,, \qquad \bar{\mathrm{D}}_{\hat{w}}F^{(w,\hat{w})} = F^{(w-1,\hat{w}+1)} \tag{B.5}$$

The laplacian on the fundamental domain of $SL(2,\mathbb{Z})$ is defined to be

$$\nabla_0^2 = 4\tau_2^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} \tag{B.6}$$

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when acting on (0,0) forms. The laplacians acting on $(w, -\hat{w})$ are defined as

$$\nabla^2_{(-)w} \equiv 4\mathrm{D}_{w-1}\bar{\mathrm{D}}_{-w} = 4\tau_2^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} - 2iw\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\bar{\tau}}\right) - w(w-1) \tag{B.7}$$

$$\nabla^2_{(+)w} \equiv 4\bar{\mathrm{D}}_{-w-1}\mathrm{D}_{w} = 4\tau_2^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} - 2iw\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\bar{\tau}}\right) - w(w+1) \tag{B.8}$$

If $\nabla^2_{(-)w} F^{(w,-w)} = \sigma_w F^{(w,-w)}$ then

$$\nabla^2_{(-)w-m} F^{(w-m,-w+m)} = (\sigma_w + 2mw - m^2 - m) F^{(w-m,-w+m)} \tag{B.9}$$

and a similar one for ∇^2_{w-m}. Using the above relations, it can be shown [3] that if as in the case of α'^3, the function $f^{(12,-12)}$ is an eigenfunction of $\nabla^2_{(-)12}$ satisfying

$$\nabla^2_{(-)12} f^{(12,-12)} = \left(-132 + \frac{3}{4}\right) f^{(12,-12)} \tag{B.10}$$

then $f^{(0,0)}$ which is the coefficient of the $\mathcal{R}^4$ term satisfies

$$\nabla^2_0 f^{(0,0)} = \frac{3}{4} f^{(0,0)} \tag{B.11}$$

The solution to the above equation is unique if $f^{(0,0)}$ has a power law behaviour near the boundary of the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ which is in agreement with known tree-level and one loop calculations. If the eigenvalue can be written as $s(s-1) > 1/4$ then the solutions are well known and can be expressed in terms of the nonholomorphic Eisenstein series

$$E_s(\tau) = \frac{1}{2}\tau_2^s \sum_{(m,n)=1} |m\tau + n|^{-2s} \tag{B.12}$$

where (m, n) denotes the greatest common divisor of m and n.

The asymptotic form of the Eisenstein series for large τ_2 can be found by manipulating the series using a Poisson resummation. The general formula for $f_s^{(0,0)}$ that is the solution for the above equation for s is

$$f^{(0,0)} = a_s \tau_2^s + b_s \tau_2^{1-s} + \frac{2\sqrt{\tau_2}\pi^s}{\Gamma(s)} \sum_{(m,n)\neq(0,0)} |\frac{m}{n}|^{s-1/2} K_{s-1/2}(2\pi\tau_2|mn|) e^{2\pi mn\tau_1} \tag{B.13}$$

where

$$a_s = 2\zeta(2s) \qquad b_s = 2\sqrt{\pi}\zeta(2s-1)\frac{\Gamma(s-1/2)}{\Gamma(s)} \tag{B.14}$$

where $K_s(x)$ is the standard modified Bessel function whose expansion for large x is given by

$$K_r(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left[\sum_{n=0}^{\infty} \frac{1}{(2x)^n)} \frac{\Gamma(r+n+1/2)}{\Gamma(r-n+1/2)\Gamma(n+1)}\right] \tag{B.15}$$

In the case we are considering, $s = 5/2$. Thus the eigenvalue in equation (B.11) is $15/4$. The coefficient of $\hat{G}^4\lambda^{16}$ is given by the solution to the equation

$$\nabla^2_{(-)14} f_2^{(14,-14)} = \left(-196 + 14 + \frac{15}{4}\right) f_2^{(14,-14)} \tag{B.16}$$

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